Lecture 7: A convenient category of spaces

In homotopy theory, it would be convenient to work with a category of spaces which has all limits, colimits, and enjoys nice properties about mapping space (especially the Exponential Law).

Compactly generated weak Hausdorff spaces give such a category **CGWH**. This will be a convenient category for homotopy theory.

Definition

A subset $Y \subset X$ is called "compactly closed" (or "k-closed") if $f^{-1}(Y)$ is closed in K for every continuous $f: K \to X$ with K compact Hausdorff. We define a new topology on X, denoted by kX, where close subsets are k-closed subsets of X. The identity

$$kX \to X$$

is a continuous map. X is called compactly generated if $kX = X$.

Note

$$k^2 X = kX.$$

Let $\mathbf{CG}$ denote the full subcategory of $\mathbf{Top}$ consisting of compactly generated spaces.

Example

Locally compact Hausdorff spaces and CW complexes are compactly generated.

Proposition

The assignment $X \to kX$ defines a functor $\mathbf{Top} \to \mathbf{CG}$, which is right adjoint to the embedding $i : \mathbf{CG} \subset \mathbf{Top}$. In other words, we have an adjoint pair

$$i : \mathbf{CG} \rightleftarrows \mathbf{Top} : k$$

Let $X \in \mathbf{CG}$, $Y \in \mathbf{Top}$. The proposition says that $f : X \to Y$ is continuous if and only if the same map $f : X \to kY$ is continuous.

Theorem

The category $\mathbf{CG}$ is complete and cocomplete. Colimits in $\mathbf{CG}$ inherit the colimits in $\mathbf{Top}$. The limits in $\mathbf{CG}$ are obtained by applying k to the limits in $\mathbf{Top}$.

Corollary

Let $\{X_i\}_{i\in I}$ be a family of objects in $\mathbf{CG}$. Then their product in $\mathbf{CG}$ is

$$k\prod_{i\in I} X_i$$

where $\prod_{i\in I} X_i$ is the topological product of X_i's (product in $\mathbf{Top}$).

We will mainly work within the category $\mathbf{CG}$ later. So we will use

$$\times, \prod \text{ to denote the categorical product in } \mathbf{CG}$$

unless otherwise specified.

Definition

Let $X, Y \in \mathbf{CG}$. We define the compactly generated topology on $\mathrm{Hom}_{\mathbf{Top}}(X, Y)$ by

$$\mathrm{Map}(X, Y) = kC(X, Y) \in \mathbf{CG}.$$

Here $C(X, Y)$ is the compact-open topology generated by

$$\{f \in \mathrm{Hom}_{\mathbf{Top}}(X, Y) | f(g(K)) \subset U\}$$

where $g : K \to X$ with K compact Hausdorff and $U \subset Y$ is open.

Note that the compact-open topology we use here for $\mathbf{CG}$ is slightly different from the usual one. We will also use the exponential notation

$$Y^X := \mathrm{Map}(X, Y).$$

Theorem

Let $X, Y, Z \in \mathbf{CG}$. Then

1. the evaluation map $\mathrm{Map}(X, Y) \times X \to Y$ is continuous;
2. the composition map $\mathrm{Map}(X, Y) \times \mathrm{Map}(Y, Z) \to \mathrm{Map}(X, Z)$ is continuous;
3. the Exponential Law holds: we have a homeomorphism

$$\mathrm{Map}(X \times Y, Z) \cong \mathrm{Map}(X, \mathrm{Map}(Y, Z)).$$

We remark that once we have a set isomorphism

$$\mathrm{Hom}_{\mathbf{Top}}(X \times Y, Z) \cong \mathrm{Hom}_{\mathbf{Top}}(X, \mathrm{Map}(Y, Z)),$$

then the fact on homeomorphism is a formal consequence.
In fact, for any $W \in \mathbf{CG}$, we have

$$\begin{aligned} \mathrm{Hom}_{\mathbf{Top}}(W, \mathrm{Map}(X \times Y, Z)) &\cong \mathrm{Hom}_{\mathbf{Top}}(W \times X \times Y, Z) \\ &\cong \mathrm{Hom}_{\mathbf{Top}}(W \times X, \mathrm{Map}(Y, Z)) \\ &\cong \mathrm{Hom}_{\mathbf{Top}}(W, \mathrm{Map}(X, \mathrm{Map}(Y, Z))). \end{aligned}$$

This implies a natural isomorphism between the two functors

$$\mathrm{Hom}_{\mathbf{Top}}(-, \mathrm{Map}(X \times Y, Z)) \cong \mathrm{Hom}_{\mathbf{Top}}(-, \mathrm{Map}(X, \mathrm{Map}(Y, Z)))$$
$$: \mathbf{CG} \to \mathbf{Set}.$$

Then Yoneda Lemma gives rise to the homeomorphism

$$\mathrm{Map}(X \times Y, Z) \cong \mathrm{Map}(X, \mathrm{Map}(Y, Z)).$$

Compactly generated weak Hausdorff space

Definition

A space X is weak Hausdorff if for every compact Hausdorff K and every continuous map $f\colon K \to X$, the image $f(K)$ is closed in X.

Let $\mathbf{wH}$ denote the full subcategory of $\mathbf{Top}$ consisting of weak Hausdorff spaces.

Let $\mathbf{CGWH}$ denote the full subcategory of $\mathbf{Top}$ consisting of compactly generated weak Hausdorff spaces.

$$\mathbf{CGWH} \subset \mathbf{CG} \subset \mathbf{Top}$$

Example

Hausdorff spaces are weak Hausdorff since compact subsets of Hausdorff spaces are closed. Therefore locally compact Hausdorff spaces are compactly generated weak Hausdorff spaces.

Example

CW complexes are compactly generated weak Hausdorff spaces.

Proposition

There exists a functor $h : \mathbf{CG} \to \mathbf{CGWH}$ which is left adjoint to the inclusion $j : \mathbf{CGWH} \to \mathbf{CG}$. That is, we have an adjoint pair

$$h : \mathbf{CG} \rightleftarrows \mathbf{CGWH} : j$$

Moreover, h preserves the subcategory $\mathbf{CGWH}$, i.e, $h \circ j$ is the identity functor.

Theorem

The category $\mathbf{CGWH}$ is complete and cocomplete. Limits in $\mathbf{CGWH}$ inherit the limits in $\mathbf{CG}$. The colimits in $\mathbf{CGWH}$ are obtained by applying h to the colimits in $\mathbf{CG}$.

Proposition

Let $X, Y \in \mathbf{CGWH}$. Then $\mathrm{Map}(X, Y) \in \mathbf{CGWH}$.

Theorem

Let $X, Y, Z \in \mathbf{CGWH}$. Then

1. the evaluation map $\mathrm{Map}(X, Y) \times X \to Y$ is continuous;
2. the composition map $\mathrm{Map}(X, Y) \times \mathrm{Map}(Y, Z) \to \mathrm{Map}(X, Z)$ is continuous;
3. the Exponential Law holds: we have a homeomorphism

$$\mathrm{Map}(X \times Y, Z) \cong \mathrm{Map}(X, \mathrm{Map}(Y, Z)).$$

Therefore $\mathbf{CGWH}$ is a full complete and cocomplete subcategory of $\mathbf{Top}$ that enjoys the Exponential Law.

Let $X \in \mathbf{CG}$ and A be a subset of X. The subspace topology on A may not be compactly generated. We equip A with a compactly generated topology by applying k to the usual subspace topology. This will be called the subspace topology in the category $\mathbf{CG}$.

When we write $A \subset X$, A is understood as s subspace of X with this compatly generated topology.

This new notion of subspace satisfies the standard characteristic property in $\mathbf{CG}$: given $Y \in \mathbf{CG}$, a map $Y \to A$ is continuous if and only if it is continuous viewed as a map $Y \to X$.

In our later discussion on homotopy theory, we will mainly work with **CGWH**. In particular, a space there always means an object in **CGWH**. All the limits and colimits are in **CGWH**. Subspace refers to the compacted generated subspace topology.

To simplify notations, we will write

$$\mathscr{T} = \mathbf{CGWH}, \quad \mathrm{h}\mathscr{T}$$

for the category **CGWH**, the quotient category of $\mathscr{T}$ by homotopy classes of maps.

We will often need the notion of a pair. Given $X, Y \in$ **CGWH**, and subspaces $A \subset X, B \subset Y$, we let

$$\mathrm{Map}((X, A), (Y, B)) = \{f \in \mathrm{Map}(X, Y) | f(A) \subset B\}$$

be the subspace of $\mathrm{Map}(X, Y)$ that maps A to B. It fits into the following pull-back diagram

$$\begin{tikzcd}
\mathrm{Map}((X, A), (Y, B)) \arrow[r] \arrow[d] & \mathrm{Map}(X, Y) \arrow[d] \\
\mathrm{Map}(A, B) \arrow[r] & \mathrm{Map}(A, Y).
\end{tikzcd}$$

We will also need the category of pointed spaces.

Definition

We define the category $\underline{\mathscr{T}_\star}$ of pointed spaces where

- an object (X, x_0) is a space $X \in \underline{\mathscr{T}}$ with a based point $x_0 \in X$
- morphisms are based continuous maps that map based point to based point

$$\mathrm{Hom}_{\underline{\mathscr{T}_\star}}((X, x_0), (Y, y_0)) = \mathrm{Map}((X, x_0), (Y, y_0)).$$

We will write

$$\mathrm{Map}_\star(X, Y) = \mathrm{Map}((X, x_0), (Y, y_0))$$

when base points are not explicitly mentioned. $\mathrm{Map}_\star(X, Y)$ is viewed as an object in $\underline{\mathscr{T}_\star}$, whose base point is the constant map from X to the base point of Y.

Theorem

$\mathscr{T}_\star$ is complete and cocomplete. Let $X, Y, Z \in \mathscr{T}_\star$. Then

1. the evaluation map $\mathrm{Map}_\star(X, Y) \wedge X \to Y$ is continuous;
2. the composition map $\mathrm{Map}_\star(X, Y) \wedge \mathrm{Map}_\star(Y, Z) \to \mathrm{Map}_\star(X, Z)$ is continuous;
3. the Exponential Law holds: we have a homeomorphism

$$\mathrm{Map}_\star(X \wedge Y, Z) \cong \mathrm{Map}_\star(X, \mathrm{Map}_\star(Y, Z)).$$

Here $\wedge$ is the smash product

$$X \wedge Y = \frac{X \times Y}{X \times \{y_0\} \cup \{x_0\} \times Y}.$$

Loop space and suspension

Definition

Let $X, Y \in \underline{\mathscr{T}_\star}$ be two pointed spaces. A based homotopy between two based maps $f_0, f_1 : X \to Y$ is a homotopy between f_0, f_1 relative to the base points. We denote $[X, Y]_0$ to be based homotopy classes of based maps. We define the category $\underline{\mathrm{h}\mathscr{T}_\star}$ by the quotient of $\underline{\mathscr{T}_\star}$ where

$$\mathrm{Hom}_{\underline{\mathrm{h}\mathscr{T}_\star}}(X, Y) = [X, Y]_0.$$

Definition

Given $(X, x_0) \in \mathscr{T}_\star$, we define the based loop space $\Omega_{x_0} X$ or simply ΩX by

$$\Omega X = \mathrm{Map}_\star(S^1, X).$$

In the unpointed case, we define the free loop space

$$\mathcal{L}X = \mathrm{Map}(S^1, X)$$

.

Theorem

The based loop space Ω defines functors

$$\Omega : \underline{\mathscr{T}_\star} \mapsto \underline{\mathscr{T}_\star}, \quad \Omega : \underline{\mathrm{h}\mathscr{T}_\star} \mapsto \underline{\mathrm{h}\mathscr{T}_\star}.$$

Proof

Let us first consider $\Omega : \underline{\mathscr{T}_\star} \mapsto \underline{\mathscr{T}_\star}$. This amounts to show that given $f : X \to Y$, the induced map

$$f_* : \mathrm{Map}_\star(S^1, X) \to \mathrm{Map}_\star(S^1, Y), \quad \gamma \to f \circ \gamma$$

is continuous. This follows since this is the same as

$$\mathrm{Map}_\star(S^1, X) \times \{f\} \to \mathrm{Map}_\star(S^1, Y).$$

Now we consider $\Omega : \underline{\mathrm{h}\mathscr{T}_\star} \mapsto \underline{\mathrm{h}\mathscr{T}_\star}$. We need to show if we have a homotopy $X \underset{g}{\overset{f}{\Downarrow}} Y$ by $F : X \times I \to Y$, then the induced maps $f_*, g_* : \mathrm{Map}_\star(S^1, X) \to \mathrm{Map}_\star(S^1, Y)$ are also homotopic.

The required homotopy is given by

$$\Omega F : \Omega X \times I \to \Omega Y, \quad (\gamma, t) \to F(-, t) \circ \gamma.$$

To see the continuity of ΩF, we first use Exponential Law to express F equivalently as a continuous map $\tilde{F} : I \to \mathrm{Map}_\star(X, Y)$.

Then ΩF is given by the composition

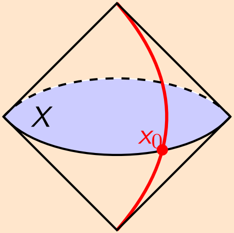
$$\mathrm{Map}_\star(S^1, X) \times I \xrightarrow{1 \times \tilde{F}} \mathrm{Map}_\star(S^1, X) \times \mathrm{Map}_\star(X, Y) \to \mathrm{Map}_\star(S^1, Y)$$

which is continuous $\square$

Definition

Let $(X, x_0) \in \underline{\mathscr{T}_\star}$. We define its suspension ΣX by

$$\Sigma X = X \times I \Big/ (X \times \partial I \cup x_0 \times I)$$



The suspension is the same as the smash product with S^1

$$\Sigma X = S^1 \wedge X.$$

Σ defines functors

$$\Sigma : \underline{\mathscr{T}_\star} \to \underline{\mathscr{T}_\star}, \quad \underline{\mathrm{h}\mathscr{T}_\star} \to \underline{\mathrm{h}\mathscr{T}_\star}.$$

Theorem

(Σ, Ω) defines adjoint pairs

$$\Sigma : \underline{\mathscr{T}_\star} \rightleftarrows \underline{\mathscr{T}_\star} : \Omega \qquad \Sigma : \underline{\mathrm{h}\mathscr{T}_\star} \rightleftarrows \underline{\mathrm{h}\mathscr{T}_\star} : \Omega$$

Group object and homotopy group

Definition

Let $\mathcal{C}$ be a category with finite product and terminal object $\star$. A group object in $\mathcal{C}$ is an object G in $\mathcal{C}$ together with morphisms

$$\mu : G \times G \to G, \quad \eta : G \to G, \quad \epsilon : \star \to G$$

such that the following diagrams commute

1) associativity:

$$\begin{tikzcd}
G \times G \times G \arrow[r, "1\times\mu"] \arrow[d, "\mu\times 1"'] & G \times G \arrow[d, "\mu"] \\
G \times G \arrow[r, "\mu"] & G
\end{tikzcd}$$

2) unit:

$$\begin{tikzcd} G\times\star \arrow[r, "1\times\epsilon"] \arrow[dr] & G\times G \arrow[d, "\mu"] & \star\times G \arrow[l, "\epsilon\times 1"'] \arrow[dl] \\ & G & \end{tikzcd}$$

3) inverse

$$\begin{tikzcd} G \arrow[r, "1\times\eta"] \arrow[d] & G\times G \arrow[d, "\mu"] & G \arrow[l, "\eta\times 1"'] \arrow[d] \\ \star \arrow[r, "\epsilon"] & G & \star \arrow[l, "\epsilon"'] \end{tikzcd}$$

μ is called the multiplication, η is called the inverse, ϵ is called the unit.

Example

Here are some classical examples.

- Group objects in **Set** are groups.
- Group objects in **Top** are topological groups.
- Group objects in **hTop** are called H-groups.

Proposition

Let $\mathcal{C}$ be a category with finite product and terminal object $\star$. Let G be a group object. Then

$$\mathrm{Hom}(-, G) : \mathcal{C} \to \mathbf{Group}$$

defines a contravariant functor from $\mathcal{C}$ to $\mathbf{Group}$.

Proof

For any $X \in \mathcal{C}$, we define the group structure on $\mathrm{Hom}(X, G)$ as:

- Multiplication: $f \cdot g = \mu(f, g)$ as

$$\begin{array}{ccccc} \mathrm{Hom}(X,G) & \times & \mathrm{Hom}(X,G) & \longrightarrow & \mathrm{Hom}(X,G) \\ X \xrightarrow{f} G & & X \xrightarrow{g} G & \mapsto & X \xrightarrow{(f,g)} G \times G \xrightarrow{\mu} G, \end{array}$$

- Inverse: $f^{-1} = \eta(f)$ as

$$\begin{array}{ccc} \mathrm{Hom}(X,G) & \longrightarrow & \mathrm{Hom}(X,G) \\ X \xrightarrow{f} G & \mapsto & X \xrightarrow{f} G \xrightarrow{\eta} G, \end{array}$$

- Identity is the image of the morphism $\mathrm{Hom}(X, \star) \to \mathrm{Hom}(X, G)$.

$\square$

Corollary

Any $X \in \underline{\mathrm{h}\mathscr{T}}$ defines a functor

$$[-, \Omega X]_0 \colon \underline{\mathrm{h}\mathscr{T}_\star} \to \underline{\mathbf{Group}}\,.$$

Definition

Let $(X, x_0) \in \underline{\mathscr{T}_\star}$. We define the n-th homotopy group

$$\boxed{\pi_n(X, x_0) = [S^n, X]_0}.$$

Sometimes we simply denote it by $\pi_n(X)$.

In particular, we have

- π_0 is the path connected component.
- π_1 is the fundamental group.
- For $n \geq 1$, we know that

$$\pi_n(X) = [\Sigma S^{n-1}, X]_0 = [S^{n-1}, \Omega X]$$

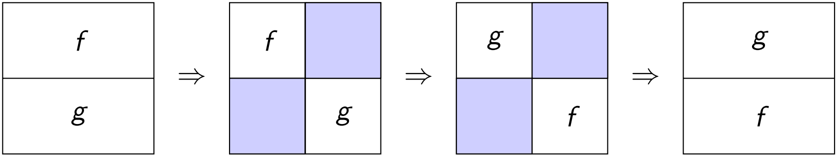
which is a group since ΩX is a group object.

Proposition

$\pi_n(X)$ is abelian if $n \geq 2$.

Proof.

This statement can be also illustrated as follows:



$\square$

Proposition

Let X be path connected. There is a natural functor

$$T_n : \Pi_1(X) \to \textbf{Group}$$

which sends x_0 to $\pi_n(X, x_0)$. In particular, there is a natural action of $\pi_1(X, x_0)$ on $\pi_n(X, x_0)$ and all $\pi_n(X, x_0)$'s are isomorphic for different choices of x_0.

Proposition

Let $f: X \to Y$ be homotopy equivalence. Then

$$f_* : \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$$

is a group isomorphism.